

Tutorial 12 - Hilbert polynomials, Bézout's theorem

20. 5. 2026

Problem 1. Compute the Hilbert function $H_I(d)$ for $d = 0, 1, 2, 3$ and the Hilbert polynomial $h_I(x)$ for the following ideals in $\mathbb{C}[x, y]$:

- $I = \langle x^2, xy \rangle$,
- $I = \langle x^3, y^2 \rangle$,

Problem 2. Let work over $\mathbb{C}[x_0, x_1, x_2, x_3]$. Compute the Hilbert polynomial for $I = \langle x_0, x_1 \rangle \cap \langle x_2, x_3 \rangle$ and $J = \langle x_0, x_2x_3 \rangle$.

Problem 3. Let $V = V(x^2 - y) \subset \mathbb{A}_{\mathbb{C}}^2$. Determine the coordinate ring $K[V]$, compute its Hilbert polynomial. Show that V is isomorphic to $\mathbb{A}_{\mathbb{C}}^1$.

Problem 4. Let $C \subset \mathbb{A}^3$ be the affine twisted cubic, parametrized by $t \mapsto (t, t^2, t^3)$. Compute the ideal $I(C)$, its Hilbert polynomial and the degree of C .

Problem 5. Let $C_1 = V(y - x^2)$ and $C_2 = V(y^2 - x^3 - x)$ in \mathbb{A}^2 . Find $C_1 \cap C_2$.

Problem 6. Let $A \subset \mathbb{A}^2$ be a finite set. Prove that there exists one line passing through exactly 2 points from A , or they all lie on the same line.

Problem 7. Prove that if $f_1, \dots, f_k \in \mathbb{F}_q[x_1, \dots, x_n]$ with $\sum_{i=1}^k \deg(f_i) < n$, then $|V(f_1, \dots, f_k)|$ is divisible by $\text{char}(\mathbb{F}_q)$.

Different flavours of algebraic geometry

Problem 8 (Projective geometry). By $\mathbb{P}^n(\mathbb{C})$ we denote the set of lines in \mathbb{C}^{n+1} without 0. Equivalently, $[x_0 : x_1 : \dots : x_n] = [\alpha x_0 : \alpha x_1 : \dots : \alpha x_n]$.

- Show that the map $(x, y) \mapsto [x : y : 1]$ embeds \mathbb{A}^2 to \mathbb{P}^2 .
- Show that every two distinct lines intersect exactly one point in \mathbb{P}^2 .
- Homogenize the following polynomials: $x^2 + y + 2$, $x^2y - 3xy + y^3$.

Problem 9 (Tropical geometry). Let $R = \mathbb{R} \cup \{+\infty\}$ with operations $a \oplus b = \min(a, b)$ and $a \otimes b = a + b$.

- Compute $3 \oplus 5$, $(4 \oplus 3) \otimes 2$.
- Determine 1 and 0.
- Write down the similar identity in R of the following: $(a + b)^2 = a^2 + 2ab + b^2$.
- Draw a curve $2 \otimes x \oplus 3 \otimes y \oplus 2$. Determine the intersection with $x \oplus 4 \otimes y \oplus 1$.

Problem 10 (Real algebraic geometry). For $f \in \mathbb{R}[x, y, z_1, \dots, z_n]$ let $V_s(f) = \{(a, b) \in \mathbb{R}^2 \mid \exists c_1, \dots, c_n : f(a, b, c_1, \dots, c_n) = 0\}$.

- Determine $V_s(x^2 - y + z^2)$, $V(x^2 + y^2 + z^2 - 1)$.
- Determine $V_s(f) \cap V_s(g)$ and $V_s(f) \cup V_s(g)$.
- For a set of solutions of $y^3 - x^2 \geq 0$, denoted as X , find f such that $X = V_s(f)$.