Tutorial 11 - Nullstellensatz and Gröbner bases

6. 5. 2026

Problem 1. Consider the following ideals $I \leq \mathbb{Z}$ and compute their radical: (0), (25), (125), (100).

Problem 2. Consider a polynomial $f = -8 + 44x - 102x^2 + 129x^3 - 96x^4 + 42x^5 - 10x^6 + x^7 \in \mathbb{C}[x]$. Compute $V(f)$ and $\sqrt{(f)}$.

Problem 3. Show that the polynomial $f(x, y) = y^2 + x^2(x-1)^2 \in \mathbb{R}[x, y]$ is irreducible, but the set $V(f) \subset \mathbb{A}^2_{\mathbb{R}}$ is reducible.

Problem 4. Determine the set $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{A}^2_{\mathbb{C}}$, decompose it to irreducible components and compute the corresponding prime ideals.

Problem 5. Prove that $\sqrt{(f)} = (\frac{f}{\mathrm{GCD}(f,f')})$ for $f \in \mathbb{C}[x]$.

Problem 6. Compute the Gröbner bases for the following ideals of $\mathbb{Q}[x, y]$:

a) $(x^2 + xy^5 + y^4, xy^6 - xy^3 + y^5 - y^2, xy^5 - xy^2, y^5 - y^2)$,
b) $(xy^3 + y^3 + 1, x^3y - x^3 + 1, x + y, y^4 - y^3 - 1)$.

Problem 7. Show that the ideals $(x+y, y^4-y^3-1)$, $(x^3y-xy^2+1, x^2y^2-y^3-1)$ and $(xy^3+y^3+1, x^3y-x^3+1, x+y)$ are all equal.

Problem 8. Find intersection of the two ideals $(x^3 - xy, y^3 - y^2 + 1)$ and $(3xy + y^2, x^2 - 2x + 1)$.