Tutorial 10 - Affine geometry

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Problem 1. Compute the ideals of the following algebraic sets in $A^2_{\mathbb{C}}$:

a) $X_1 = \{(1,0),(0,1)\}$
b) $X_2 = \{(1,0),(0,1),(0,0)\}$
c) $X_3 = \{(1,0),(0,1),(\frac{1}{2},\frac{1}{2})\}$

What is the minimal number of polynomials you need to generate $I(X_1)$, $I(X_2)$ and $I(X_3)$ respectively?

Problem 2. Let X be the union of the coordinate axes in A^n. Find generators for the ideal of X. How many polynomials do you need to generate $I(X)$?

Problem 3. Show that a finite union, as well as an arbitraty intersection, of varietes is a variety. Conclude that varietes define a topology.

Problem 4. Prove that sets $\mathbb{Z} \subset \mathbb{R}$ and $[0,1]^2 \subset \mathbb{R}^2$ are not algebraic varietes over $\mathbb{R}$.

Problem 5. For every n, find an ideal in $\mathbb{R}[x,y]$ that needs at least n generators.

Problem 6. Show that for every n there exist n algebraically independent real numbers.

Problem 7. Show that if $a_1, a_2, \dots a_N \in \mathbb{R}^n$ are points whose nN coordinates are algebraically independent, and if $N = \binom{d+n}{n}$ then the only polynomial $f \in \mathbb{R}[x_1, \dots x_n]$ of degree at most d vanishing at all the a_i is the zero polynomial.

Definition 8. Given sets X, Y and relation $R \subseteq X \times Y$, define maps

$$A^{\rightarrow} = \{y \in Y : \forall x \in A, xRy\}; B^{\leftarrow} = \{x \in X : \forall y \in B, xRy\}$$

Then $A \to A^{\rightarrow}$ and $B \to B^{\leftarrow}$ form a *Galois connection.*

We say that a set is *closed* if $A = (A^{\rightarrow})^{\leftarrow}$ (or similarly $B = (B^{\leftarrow})^{\rightarrow}$).

Problem 9. Let X be a set and R be the binary relation on 2^X defined by

$$(U,V) \in R \iff U \cap V \neq \emptyset$$

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Consider Galois correspondence on sets 2^{2^X} and 2^{2^X} induced by this relation.

a) Let $X = \{1,2,3,4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \{\{1,2\},\{2,3\}\}$ and $A = \{\{1,2\},\{2\}\}$.

b) Prove that if Galois correspondence is defined by a symmetric relation a set, then the closure operators induced by it coincide.

c) Prove that for every $A \subseteq 2^X$ we have $A^{\rightarrow\leftarrow} = \{U \in 2^X : \exists V \in A, V \subseteq U\}$.

Problem 10. Consider $X = Y$ is a vector space of finite dimension with inner product and $(v,w) \in R$ iff $v \perp w$. Show that closed sets are subspaces.