# Tutorial 5 - Continuous Fourier transform

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**Problem 1.** Find all characters of $\mathbb{T}$ and verify that $\widehat{\mathbb{T}} \cong \mathbb{Z}$.

**Problem 2.** Let $G$ be an abelian group and $\widehat{G}$ its dual group. For every $a \in G$ we define $f_a \colon \widehat{G} \to \mathbb{T}$ by $f_a(\chi) = \chi(a)$. Verify that:

a) $f_a$ is a group homomorphism,

b) the mapping $a \mapsto f_a$ is a group homomorphism.

**Problem 3.** Let $f \colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a piecewise smooth function. Prove that $S_n(f) = f * D_n$.

**Problem 4.** Prove that $D_n(x) = \frac{\sin((2n+1)\pi x)}{\sin(\pi x)}$.

**Problem 5.** Let $f(x) = \operatorname{sgn}(x)$, $x \in (-\pi, \pi]$. Expand $f$ to the Fourier series and determine the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}$.

**Problem 6.** Let $f(x) = \pi^2 - x^2$, $x \in (-\pi, \pi]$. Expand $f$ to the Fourier series and determine the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

**Problem 7.** Consider the function of the real variable $f(x) = e^{-\pi x^2}$. Prove that the Fourier transform satisfies $\widehat{f}(a) = f(a)$ for all $a \in \mathbb{R}$.

**Problem 8.** Find all 1-periodic functions $g$ that satisfies the following differential equation:

$$g''(x) + 2g'(x) + g(x) = \sin 2\pi x.$$