Tutorial 4 - Convolution and Gauss sum

11. 3. 2026

Problem 1. Verify that for functions $f, g, h \in \mathbb{C}G$ and $a, b \in \mathbb{C}$, the convolution satisfies:

a) $f * g = g * f$,

b) $f * (g * h) = (f * g) * h$,

c) $(af + bg) * h = a(f * h) + b(g * h)$.

Problem 2. Prove $\widehat{fg}(a) = \frac{1}{|G|}(\widehat{f} * \widehat{g})(a)$.

Problem 3. Prove that $||f * g||_1 \leq ||f||_1 \cdot ||g||_1$.

Problem 4 (Hölder inequality)**.** Let $a, b \in \mathbb{C}$. Prove that $||ab||_1 \leq ||a||_p ||b||_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 5. Let $H \leq G$ be finite abelian groups. Then show that

$$\widehat{1_H}(a) = \begin{cases} \frac{|H|}{|G|} & \text{if } a \in H^\perp; \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6. Let $H \leq G$ be finite abelian groups. Apply Poisson summation formula to prove that $|G| = |H| \cdot |H^\perp|$.

Problem 7. Let p be a prime. For $x \in \mathbb{Z}_p$, we define the **Legendre symbol** as:

$$\left(\frac{x}{p}\right) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{p}, \\ 1 & \text{if } x \text{ is a nonzero quadratic residue modulo } p, \\ -1 & \text{otherwise.} \end{cases}$$

a) Show for odd primes that $\left(\frac{-1}{p}\right) = 1$ if and only if $p \equiv 1 \pmod 4$.

b) Show that Legendre symbol is multiplicative, i.e. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$.

c) Prove that $\left(\frac{x}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.

d) Show that $\widehat{\left(\frac{x}{p}\right)} = c \cdot \left(\frac{x}{p}\right)$ for a suitable constant c.

Problem 8. Let p be a prime number.

a) Prove that $\mathrm{Gau}(rs^2) = \mathrm{Gau}(r)$ for $s \in \mathbb{Z}_p^*$.

b) Verify that if -1 is not a quadratic residue in $\mathbb{Z}_p$, then $\mathrm{Gau}(-r) = -\mathrm{Gau}(r)$.

c) Let $G = \mathbb{Z}_p^2$ and define $f\colon G \to \mathbb{C}$, $f(x, y) = e^{2\pi i (x^2+y^2)\frac{r}{p}}$. Let $H = \mathbb{Z}_p \oplus \{0\} \leq G$. Apply Poisson summation formula to prove $\mathrm{Gau}(1)^2 = \pm p$.