Tutorial 3 - Applications of Fourier analysis

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Problem 1. Let us define **correlation** of two boolean functions $f, g \in \mathbb{C}\mathbb{Z}_2^n$ as:

$$C(f,g) = 2\mathbb{P}[f = g] - 1 = 2\frac{|\{u \in \mathbb{Z}_2^n : f(u) = g(u)\}|}{2^n} - 1.$$

Prove that:

a) $C(f,g) \in [-1,1]$,
b) $C(f,g) = C(g,f)$,
c) $C(f,g) = C(f+g,0)$,
d) $C(f,g) = 1 \iff f = g$,
e) $C(f,g) = \langle \widehat{f}, \widehat{g} \rangle$, where $\widehat{f}(x) = (-1)^{f(x)}$ is called **sign function** of f.

Problem 2. We define a **parity function** p_v as $p_v(u) = v \cdot u$. Show that $\widehat{f} = \sum C(f, p_v) \cdot \widehat{p_v}$. Think out that we basically perform Fourier analysis of boolean functions.

Problem 3. Consider function $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^m$. We define a **correlation matrix** $C^f \in M(\mathbb{R})_{2^m \times 2^n}$ as $C^f_{u,v} = C(p_u \circ f, p_v)$. Consider a map $L_n\colon \mathbb{Z}_2^n \to \mathbb{R}^{2^n}$ as $v \mapsto \widehat{p_v}$.

a) Show that $L_m \circ f = C^f \circ L_n$.
b) Show that for functions $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^m$, $g\colon \mathbb{Z}_2^m \to \mathbb{Z}_2^k$ it holds that $C^{g \circ f} = C^g \cdot C^f$.
c) Show that if $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is a bijection and $g, h \in \mathbb{C}\mathbb{Z}_2^n$, then $C(g \circ f, h) = C(g, h \circ f^{-1})$.

Problem 4 (Pilling-up lemma)**.** For $f_1, f_2, \dots, f_k \in \mathbb{C}\mathbb{Z}_2^n$ pairwise independent random variables it holds that

$$C(f_1 + f_2 + \dots + f_k, 0) = \prod_{i=1}^{k} C(f_i, 0).$$

Problem 5 (Gowers U^2 uniformity norm)**.** Let $f\colon \mathbb{F}_p^n \to \mathbb{C}$. Define:

$$||f||_{U^2} = (\mathbb{E}[f(x)\overline{f(x+y)f(x+z)}f(x+y+z)])^{\frac{1}{4}}.$$

a) Show that it is well defined. Also show that $||f||_{U^2} \geq |\mathbb{E}[f]|$.
b) For $f_i\colon \mathbb{F}_p^n \to \mathbb{C}$, let

$$\langle f_1, f_2, f_3, f_4 \rangle = \mathbb{E}[f_1(x)\overline{f_2(x+y)f_3(x+z)}f_4(x+y+z)].$$

Prove that:

$$|\langle f_1, f_2, f_3, f_4 \rangle| \leq ||f_1||_{U^2} ||f_2||_{U^2} ||f_3||_{U^2} ||f_4||_{U^2}.$$

c) Show that

$$||f+g||_{U^2} \leq ||f||_{U^2} + ||g||_{U^2}$$

and conclude that $||\cdot||_{U^2}$ is a norm.

d) Show that $||f||_{U^2} = ||\widehat{f}||_4$. Also if $||f||_\infty \leq 1$, then $||\widehat{f}||_\infty \leq ||f||_{U^2} \leq ||\widehat{f}||_\infty^{\frac{1}{2}}$.

Problem 6 (4-AP counts)**.** Let $A = \{x \in \mathbb{F}_5^n : x \cdot x = 1\}$. Prove that:

a) $|A| = (\frac{1}{5} + o(1))5^n$ and $|\widehat{1_A}(r)| = o(1)$ for all $r \neq 0$.
b) $|\{(x,y) \in \mathbb{F}_5^n : x, x+y, x+2y, x+3y \in A\}| \neq (5^{-4} + o(1))5^{2n}$.