

# Tutorial 2 - Fourier analysis

18. 2. 2026

**Problem 1.** Let  $S \subseteq [n]$  and  $\mathbb{Z}_2^n$  be a group.

- Prove that the map  $\chi_S(x) = (-1)^{\sum_{i \in S} x_i}$  is a character  $\mathbb{Z}_2^n \rightarrow \mathbb{C}^*$ ,
- for  $T \subseteq [n], S \neq T$ , show that  $\chi_S \neq \chi_T$ .

**Problem 2.** Generalize the previous problem into general case of finite abelian groups.

**Problem 3.** Let  $G$  be a finite abelian group and  $a \in G$ . Then compute the Fourier transform of:

- $\chi_a$ ,
- $\delta_a$ ,
- $\delta_{-1} + \delta_0 + \delta_1$  for  $G = \mathbb{Z}_n$ ,
- $\sum_{i=0}^{k-1} \delta_{il}$  for  $G = \mathbb{Z}_{kl}$ .

**Problem 4.** Verify that:

- $\widehat{f}(0) = \mathbb{E}[f]$  for every finite abelian group  $G$  and every function  $f: G \rightarrow \mathbb{C}$ ,
- $\|\widehat{f}\|_\infty \leq \|f\|_1$ .

**Problem 5.** Fix  $p \in G$  and  $c \in \mathbb{C}$ . We define operators  $T_p$  and  $P_c$  over  $\mathbb{C}G$  such that for every function  $f: G \rightarrow \mathbb{C}$  it holds that  $(T_p f)(x) = f(x+p)$ ,  $(P_c f)(x) = cf(x)$ .

In case  $G$  is a field and  $p \neq 0$ , we define  $(S_p f)(x) = f(px)$ . Prove that:

- $\widehat{T_p f}(a) = \chi_a(p) \widehat{f}(a)$ ,
- $\widehat{P_c f}(a) = c \widehat{f}(a)$ ,
- $\widehat{S_p f}(a) = \widehat{f}\left(\frac{a}{p}\right)$ .

**Problem 6.** Consider group  $\mathbb{Z}_n$  and matrix  $M \in \mathbb{C}_{n \times n}$  such that for every  $f \in \mathbb{Z}_n \rightarrow \mathbb{C}$  we have:

$$(\widehat{f}(0), \widehat{f}(1), \dots, \widehat{f}(n-1))^T = M(f(0), f(1), \dots, f(n-1))^T.$$

Compute  $\det M$  and prove that Fourier transform is a bijection.

**Problem 7.** Draw a Cayley graph of  $\mathbb{Z}_6$  with respect to the set  $S = \{\pm 2, \pm 3\}$  and compute the eigenvalues of the adjacency matrix.