Exercise 7 – UFD and Euclidean domains

November 18, 2026

Problem 1. Find the irreducible factorizations of the elements 3, 5, 6, and $10 - 6i$ in $\mathbb{Z}[i]$.

Problem 2. Determine whether the number 6 has more than one factorization in each of the rings $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-5}]$, and $\mathbb{Z}[\sqrt{6}]$. If so, find all possible factorizations.

Problem 3. Perform the following divisions with remainder:

a) Divide $5 + 7i$ by $3 - i$ in $\mathbb{Z}[i]$.

b) Divide $3 + 2i$ by $1 + i$ in $\mathbb{Z}[i]$.

c) Divide 4 by $1 - \sqrt{-2}$ in $\mathbb{Z}[\sqrt{-2}]$.

d) ◁ Divide $1 + 4\sqrt{-2}$ by $3 + \sqrt{-2}$ in $\mathbb{Z}[\sqrt{-2}]$.

e) ◁ Divide $5 + 2\omega$ by $1 - \omega$ in $\mathbb{Z}[\omega]$, where $\omega = e^{\frac{2\pi i}{3}}$.

Problem 4. Compute the greatest common divisor (GCD) of the following pairs and, if possible, determine their Bézout coefficients:

a) $x^3 - 7x^2 + 15x - 9$ and $x^3 - 7x^2 + 14x - 8$ in $\mathbb{Z}[x]$.

b) 5 and $7 + i$ in $\mathbb{Z}[i]$.

c) 3 and $2 + \sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$.

d) $x^3 + x^2 + x + 1$ and $x^2 - 2x + 2$ in $\mathbb{Z}_3[x]$.

e) $6 - 3\sqrt{3}$ and $3 + \sqrt{3}$ in $\mathbb{Z}[\sqrt{3}]$.

Problem 5. Explain the following phenomenon:

- In the ring $\mathbb{Z}[\sqrt{-3}]$, we have
$$2(-2) = (\sqrt{-3} + 1)(\sqrt{-3} - 1),$$
and therefore this ring does not have unique factorization.
- In the ring $\mathbb{Z}[\sqrt{2}]$, we have
$$2 = (-4 + 3\sqrt{2})(4 + 3\sqrt{2}),$$
yet this ring does have unique factorization.

Problem 6. Factor the polynomial $2x^2 + 2x - 1$ over the Euclidean domain $\mathbb{Z}[\sqrt{3}]$ into a product of irreducible elements.

Problem 7. Show that the polynomial
$$3x^3 + 2x^2 + (4 - 2i)x + (1 + i)$$
is irreducible in $\mathbb{Z}[i][x]$.

Homework. Find infinitely many invertible elements in the ring $\mathbb{Z}[\sqrt{3}]$.