Exercise 5 – Irreducibility

November 4, 2026

Problem 1. Find the irreducible factorizations of the following polynomials in the integral domains $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Q}[x]$, $\mathbb{Z}_3[x]$, and $\mathbb{Z}_5[x]$:

a) $x^3 - 2$,

b) $x^4 - x^2 - 2$.

Problem 2. ◁ Show that every irreducible polynomial in $\mathbb{R}[x]$ has degree at most 2.

Problem 3. Find all irreducible polynomials in $\mathbb{Z}_2[x]$ of degree at most 3.

Problem 4. ◁ Show that, for every $t \in \mathbb{Q}$ and $f \in \mathbb{Q}[x]$, the polynomial $f(x)$ is irreducible in $\mathbb{Q}[x]$ if and only if $f(x+t)$ is irreducible in $\mathbb{Q}[x]$.

Problem 5. Find the irreducible factorizations of the elements 3, 5, 6, and $10 - 6i$ in the integral domain $\mathbb{Z}[i]$.

Problem 6. Determine whether the element 6 has more than one factorization into irreducibles in each of the integral domains $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{-5}]$, and $\mathbb{Z}[\sqrt{6}]$. If so, try to find all such factorizations.

Problem 7. Compute the greatest common divisors (GCDs) of the following pairs:

a) $x^3 - 7x^2 + 15x - 9$ and $x^3 - 7x^2 + 14x - 8$ in $\mathbb{Z}[x]$,

b) 5 and $7 + i$ in $\mathbb{Z}[i]$,

c) 3 and $2 + \sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$,

d) $x^3 + x^2 + x + 1$ and $x^2 - 2x + 2$ in $\mathbb{Z}_3[x]$,

e) $6 - 3\sqrt{3}$ and $3 + \sqrt{3}$ in $\mathbb{Z}[\sqrt{3}]$.

Problem 8. Explain the following phenomenon:

- In the integral domain $\mathbb{Z}[\sqrt{-3}]$, we have

  $$2(-2) = (\sqrt{-3} + 1)(\sqrt{-3} - 1),$$

  and hence it is not a unique factorization domain.

- In the integral domain $\mathbb{Z}[\sqrt{2}]$, we have

  $$2 = (-4 + 3\sqrt{2})(4 + 3\sqrt{2}),$$

  even though it is a unique factorization domain.

Problem 9. For an integral domain D, let $\mathrm{U}(D)$ denote the set of all units (invertible elements) of D. Determine $\mathrm{U}(\mathbb{Z})$, $\mathrm{U}(\mathbb{Z}[i])$, $\mathrm{U}(\mathbb{Z}[\sqrt{2}])$, and $\mathrm{U}(\mathbb{C}[x])$.

Problem 10. Determine $\mathrm{U}(\mathbb{Z}[\omega])$, where $\omega = e^{\frac{2\pi i}{3}}$ is a root of the polynomial $x^2 + x + 1$. Plot the elements of $\mathbb{Z}[\omega]$ in the complex plane and indicate the set $\mathrm{U}(\mathbb{Z}[\omega])$.

Homework. Show that, for every natural number $n \in \mathbb{N}$ that is not a perfect square, the set of units $\mathrm{U}(\mathbb{Z}[\sqrt{-n}])$ is equal to $\{1, -1\}$.

Hint: use norm.