

Exercise 4 – Polynomials

October 21, 2026

Problem 1. Divide the following pairs of polynomials with remainder:

- a) $x^4 + 3x^3 + 4x^2 + x + 3$ and $x^2 + 2$ in $\mathbb{Z}[x]$,
- b) $x^4 + 3x^3 + 4x^2 + x + 3$ and $x^2 + 2$ in $\mathbb{Z}_5[x]$,
- c) $x^{10} + x^9 + x^7 + x^5 + x^3 + x^2 + x$ and $x + 1$ in $\mathbb{Z}_2[x]$,
- d) $x^n - 1$ and $x^m - 1$ in the ring $\mathbb{Z}[x]$ for $n \geq m \geq 1$.

Problem 2. Prove that $x^m - 1 \mid x^n - 1$ in $\mathbb{Z}[x]$ if and only if $m \mid n$, for all $m, n \in \mathbb{N}$.

Problem 3. Show that there are infinitely many distinct polynomials in $\mathbb{Z}_2[x]$ that define the same polynomial function $m: \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$, where $m(a) = a$ for every $a \in \mathbb{Z}_2$.

Problem 4. Find all roots of the following polynomials in the indicated rings:

- a) $x^2 + 2x + 3$ in $\mathbb{Z}_5$,
- b) $x^4 - 5x^2 + 4$ in $\mathbb{Z}$,
- c) $x^2 + 4$ in $\mathbb{Z}_{12}$,
- d) $x^2 - 7$ in $\mathbb{Z}_{15}$,
- e) $x^2 - 2\sqrt{2}x + 2$ in $\mathbb{Z}[\sqrt{2}]$.

Problem 5. For a polynomial $f \in \mathbb{C}[x]$, let $V(f)$ denote the set of all roots of f . Consider the polynomial $f = -8 + 44x - 102x^2 + 129x^3 - 96x^4 + 42x^5 - 10x^6 + x^7 \in \mathbb{C}[x]$. Determine $V(f)$ and find $g \in \mathbb{C}[x]$ such that $V(g) = V(f)$ and g has the smallest possible degree. Derive a general rule.

Problem 6. Find a polynomial of degree exactly one in $\mathbb{Z}_4[x]$ that has no roots in $\mathbb{Z}_4$.

Problem 7. Let D be an integral domain. Determine the conditions under which a polynomial $a_1x + a_0 \in D[x]$, with $a_1 \neq 0$, has a root in D .

Problem 8. Show that there is no $m \in \mathbb{N}$ such that $x^2 - 1$ has exactly 3 distinct roots in $\mathbb{Z}_m$.

Problem 9. Let R be a ring. Determine all invertible polynomials in $R[x]$ under the following assumptions:

- a) R is a field,
- b) R is an integral domain.

Problem 10. Find a ring R and a nonzero polynomial f such that f has infinitely many distinct roots in R .

Homework. Show that for every finite field T , there exists a nonconstant polynomial $f \in T[x]$ that has no roots in T .